# Modern approaches to quantum gravity

## Homework 9 — Fall 2025

### 1. Basics of 2D CFT

When working with tensors in $d = 2$, it can be useful to work in coordinates $z = x + iy$ and $\bar{z} = x - iy$.

(a) Let $M_{\mu_1 \dots \mu_\ell}$ be a traceless symmetric tensor of spin $\ell = 0, 1, 2, \dots$. Show that all independent components of $M$ can be expressed in terms of $M := M_{z \dots z}$ and $\bar{M} := M_{\bar{z} \dots \bar{z}}$. How do $M, \bar{M}$ transform under rotations? If $M_{\mu_1 \dots \mu_\ell}$ is conserved, show that $M$ and $\bar{M}$ are holomorphic and antiholomorphic respectively

(b) Apply this to a conserved vector $J_\mu$ and the stress tensor $T_{\mu\nu}$. What is the most general form of the 2-pt functions

$$\langle J(z,\bar{z}) J(w,\bar{w}) \rangle, \quad \langle J(z,\bar{z}) \bar{J}(w,\bar{w}) \rangle, \quad \langle \bar{J}(z,\bar{z}) \bar{J}(w,\bar{w}) \rangle \tag{1}$$

and likewise for $T, \bar{T}$? What are the consequences of parity invariance, which we can choose to send $(x, y) \mapsto (x, -y)$?

(c) Using your previous results, we can write

$$T(z) = \sum_{n \in \mathbb{N}} \frac{L_n}{z^{2+n}} \quad \text{and} \quad \bar{T}(z) = \sum_{n \in \mathbb{N}} \frac{\bar{L}_n}{\bar{z}^{2+n}}. \tag{2}$$

After a tedious computation, it can be shown that the modes satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m) L_{m+n} + \frac{c}{12} n(n^2 - 1) \delta_{m+n,0} = [\bar{L}_n, \bar{L}_m], \quad [L_n, \bar{L}_m] = 0 \tag{3}$$

for some theory-dependent coefficient $c$. Show that the modes with labels $m, n \in \{-1, 0, 1\}$ form a subalgebra. Identify the $3 + 3 = 6$ generators in this subalgebra with the 6 generators of the conformal group $SO(3,1)$.

(d) In radial quantization, we put the CFT on the cylinder $\mathbb{R} \times S^{d-1} = \mathbb{R} \times S^1$, and local operators map one-to-one to states in the Hilbert space. The $\{L_n, \bar{L}_n\}$ act on the Hilbert space, too. Suppose that a state obeys

$$L_0 |O\rangle = h |O\rangle, \quad \bar{L}_0 |O\rangle = \bar{h} |O\rangle. \tag{4}$$

What is the interpretation of $h, \bar{h}$? Recall that a primary state of the conformal algebra obeys $K_\mu |O\rangle = 0$, by definition. What does this translate to in the Virasoro language?

(e) A state is called a Virasoro primary if it obeys $L_n|O\rangle = 0$ for all $n = 1, 2, \dots$. Suppose that $|O\rangle$ is indeed a Virasoro primary and that it satisfies 4.
In what follows, you may use the fact that in radial quantization, the modes obey $(L_n)^\dagger = L_{-n}$. Compute the Gram matrix $G_{ij}$ of the states $\{L_{-1}|O\rangle, L_{-2}|O\rangle, (L_{-1})^2|O\rangle\}$, which is a $3\times 3$ matrix. Unitarity requires that $G$ is positive-semidefinite, what does this imply for $h$ and $c$? Consider in particular the case of the unit operator $O = 1$, which by construction has $h = \bar{h} = 0$ and exists in any CFT.
Re-prove the 2d case of the unitarity bounds from the previous problem set.
*Note:* consider in particular the possibility that $G_{ij}$ has a null state, that is to say there exists a vector $v^i$ such that $G_{ij}v^j = 0$. In that case, we can quotient out this null state.

## 2. *(Optional)* OPE and holomorphicity for free scalars

(a) Using the divergence theorem, verify that

$$\partial\bar{\partial}\ln|z|^2 = 2\pi\delta(z,\bar{z}) \tag{5}$$

(b) Consider a theory of $D$ free scalar fields $X^\mu(z,\bar{z})$ whose OPE is given by

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{6}$$

Suppose $X^\mu$ has a mode expansion with $\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1}$. Show that the modes are recovered as

$$\alpha_n^\mu = i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}z^n\partial X^\mu(z). \tag{7}$$

Hence show that the OPE implies the usual commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{8}$$

for these modes.

(c) The holomorphic stress tensor of this free scalar theory is

$$T(w) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : (w). \tag{9}$$

By considering the OPE of $X^\mu(z)$ with $T(w)$, prove that $\partial^n X^\mu(z)$ has definite conformal weight $(h,\bar{h}) = (n, 0)$, but that it is a primary operator only when $n = 1$.

## 3. Weyl transformations are anomalous in $d = 2$

Consider the partition function $Z[g_{\mu\nu}]$ of a CFT on some metric $g_{\mu\nu}$. In principle, we have $Z[\Omega^2 g] = Z[g]$, since CFTs are insensitive to Weyl transformations. However, in two-dimensional CFTs, this transformation is anomalous, and we have

$$Z[g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}] = Z[\delta_{\mu\nu}] \exp\left\{ -\frac{c}{24\pi} \int d^2x\, \sigma(x)\delta_{\mu\nu}\partial_\mu\partial_\nu\sigma(x) \right\}, \tag{10}$$

for some theory-dependent coefficient $c$. Using this action, $n$-point functions of $T_{\mu\nu}$ can be computed in any background; they are universal and can only depend on $c$.

(a) Show that in the coordinates $z, \bar{z}$,

$$\langle T_{zz} \rangle_g = \frac{c}{12\pi} \left( \partial_z^2 \sigma - (\partial_z \sigma)^2 \right). \tag{11}$$

**Hint:** Use the Weyl anomaly $g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{c}{24\pi} R$ and the conservation $\nabla^\mu \langle T_{\mu\nu} \rangle = 0$. Remember that in complex coordinates, $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^{2\sigma}$ and $g^{z\bar{z}} = g^{\bar{z}z} = 2e^{-2\sigma}$, with all other components vanishing.

(b) The cylinder $M = \mathbb{R} \times S^1$ is related to $\mathbb{R}^2$ by a Weyl transformation. Compute the corresponding $\sigma$ and compute the one-point functions $\langle T_{\mu\nu} \rangle_M$. How do you interpret this result?

(c) Verify that in 2 dimensions, the conformal Killing vectors are generated by all holomorphic changes of coordinates: $w = f(z)$, $\bar{w} = \bar{f}(\bar{z})$. Under a conformal transformation, the stress tensor changes like a primary, up to an inhomogeneous piece due to the anomaly:

$$T'_{ww}(w) = \left( \frac{dz}{dw} \right)^2 \left( T_{zz} + \frac{c}{24\pi} \{w, z\} \right). \tag{12}$$

The term $\{w, z\}$ is called the Schwartzian derivative and reads

$$\{w, z\} = \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2} \left( \frac{\partial_z^2 w}{\partial_z w} \right)^2. \tag{13}$$

Derive (13) using the result from point (a). *Hint:* choose a Weyl transformation that compensates for the diffeomorphism $w = f(z)$, $\bar{w} = \bar{f}(\bar{z})$.

## 4. Coordinates and Isometries of AdS

(a) Check that the following metrics each describe a $D$-dimensional anti-de Sitter spacetime with unit radius (or a subset or quotient thereof), by identifying coordinate transformations which relate them to the Embedding space coordinates (or to each other).

$$\text{Embedding space:} \quad ds^2 = \eta_{AB} dX^A dX^B, \text{ with: } \eta_{AB} X^A X^B = -1; \tag{14}$$
$$\text{Global:} \quad ds^2 = d\rho^2 - \cosh(\rho)^2 dt^2 + \sinh(\rho)^2 d\Omega_{D-2}^2; \tag{15}$$
$$\text{Global 2:} \quad ds^2 = -(1+r^2)dt^2 + (1+r^2)^{-1} dr^2 + r^2 d\Omega_{d-1}^2; \tag{16}$$
$$\text{Conformal:} \quad ds^2 = \frac{1}{\cos(\theta)^2}\left[-dt^2 + d\theta^2 + \sin(\theta)^2 d\Omega_{D-2}^2\right]; \tag{17}$$
$$\text{Poincaré patch:} \quad ds^2 = \frac{1}{z^2}\left[-dt^2 + dz^2 + \delta_{ij} dx^i dx^j\right], \text{ with: } z > 0; \tag{18}$$
$$\text{Cosmological:} \quad ds^2 = -dT^2 + \cos(T)^2\left[dr^2 + \sinh(r)^2 d\Omega_{D-2}^2\right]. \tag{19}$$

Here $\eta_{AB}$ is the flat metric in a $D-$dimensional spacetime with signature $(2, D-1)$.

*Note 1:* AdS in embedding space contains some closed timelike curves. To remove this feature, one normally considers the universal covering of AdS. For example, this is achieved in global coordinates by allowing $t \in \mathbb{R}$ instead of $t \in [0, 2\pi]$.

*Note 2:* Equation (19) resembles a recollapsing FLRW cosmology with $k = -1$ (negative curvature), but the apparent 'Big Bang' and 'Big Crunch' singularities are actually just coordinate artifacts. You may find it illuminating to sketch the patch of AdS which this geometry covers on the 'tin can' diagram.

(b) *(Optional)* Draw the Penrose diagram of AdS spacetime. Draw the portion of the full diagram that is covered by each of the coordinates above.

(c) Which symmetry group is manifest in each of the coordinates above?

(d) Identify the $D(D+1)/2$ Killing vectors of AdS which satisfy Killing's equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0,$$

by writing them down explicitly in AdS-Poincaré coordinates (18).

(e) Show that on the conformal boundary ($z = 0$), these Killing vectors reduce to the $(d+1)(d+2)/2$ conformal Killing vectors on Minkowski spacetime.